

INTERACTIVE OPTIMIZATION & LEARNING
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Out-of-the-Box Global Optimization for Packing Problems

New models and improved solutions for hard geometric packings

Gioni Mexi

joint work with Timo Berthold, Dominik Kamp, Sebastian Pokutta, Imre Pólik

Zuse Institute Berlin

IFORS 2026, Vienna



Motivation: AI or Optimization?

What if you just ask an LLM agent?

Task: pack 26 circles in a unit square maximizing the sum of their radii

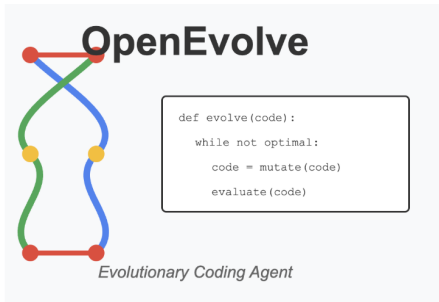


Image source:

<https://github.com/algorithmicsuperintelligence/openevolve>

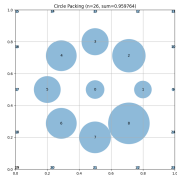
... openevolve · system_prompt

*"You are an expert mathematician specializing in circle packing... improve a constructor that arranges 26 circles in a unit square, **maximizing the sum of their radii**. AlphaEvolve achieved 2.635."*

phase 2, after plateauing at 2.377

*"... the **optimization routine is critically important**... try fundamentally different approaches — **don't just tweak parameters**."*

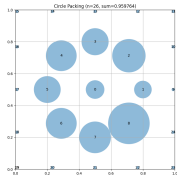
... and it evolved its way to *optimization*



Gen 1 — rings

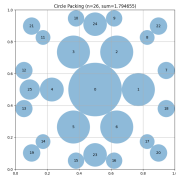
$$\Sigma r = 0.96$$

... and it evolved its way to *optimization*



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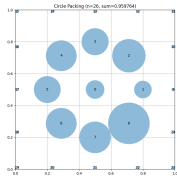


Gen 10 — hexagonal

$$\Sigma r = 1.80$$

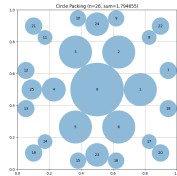


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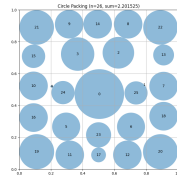
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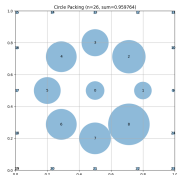


Gen 100 — grid

$$\Sigma r = 2.20$$

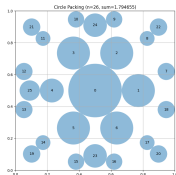


... and it evolved its way to *optimization*



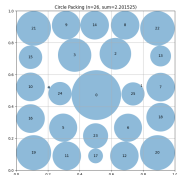
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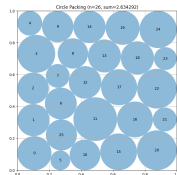
Gen 10 — hexagonal

$$\Sigma r = 1.80$$



Gen 100 — grid

$$\Sigma r = 2.20$$



Gen 460 — SLSQP

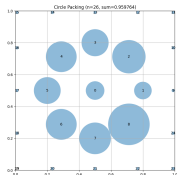
$$\Sigma r = 2.63$$

```
from scipy.optimize import minimize
result = minimize(objective, x0, method='SLSQP',
                  bounds=bounds,
                  constraints=cons)
```



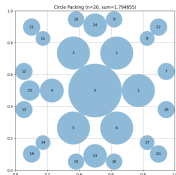
Use mathematical optimization!

... and it evolved its way to *optimization*



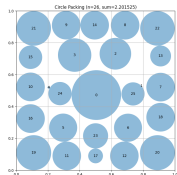
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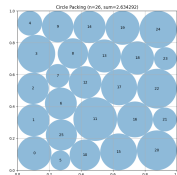
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Use mathematical optimization!

So let's do that on purpose

Studied packing problems

1. Circles in a square / fixed-perimeter rectangle
2. Circles in a minimum-area ellipse
3. Regular polygons in a regular polygon
4. Platonic solids in a Platonic solid (first systematic study)

Contributions

- ◇ New formulations for circle, ellipse, polygon, and Platonic-solid packing
- ◇ New best-known solutions for many packing problems, including first solutions for some variants
- ◇ Insight: Global optimization solvers such as **SCIP** and **FICO Xpress** have become very capable (Berthold et al., 2026b,a)

- ◇ Research framework: plugin-based — implement and test your own heuristics, separators, propagators, ...
- ◇ basis for specialized extensions (GCG, SCIP-SDP, SCIP-Jack, ReXi, ...)



<https://www.scipopt.org/>

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- ◇ basis for specialized extensions (GCG, SCIP-SDP, SCIP-Jack, ReXi, ...)
- ◇ A high-performance open-source solver for mixed-integer linear and nonlinear programs
- ◇ open source (Apache 2.0)



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<https://www.scipopt.org/>

Best open-source MIP solver for quick primal solutions

new **concurrent mode** (SCIP 11): racing portfolio with solution & bound sharing

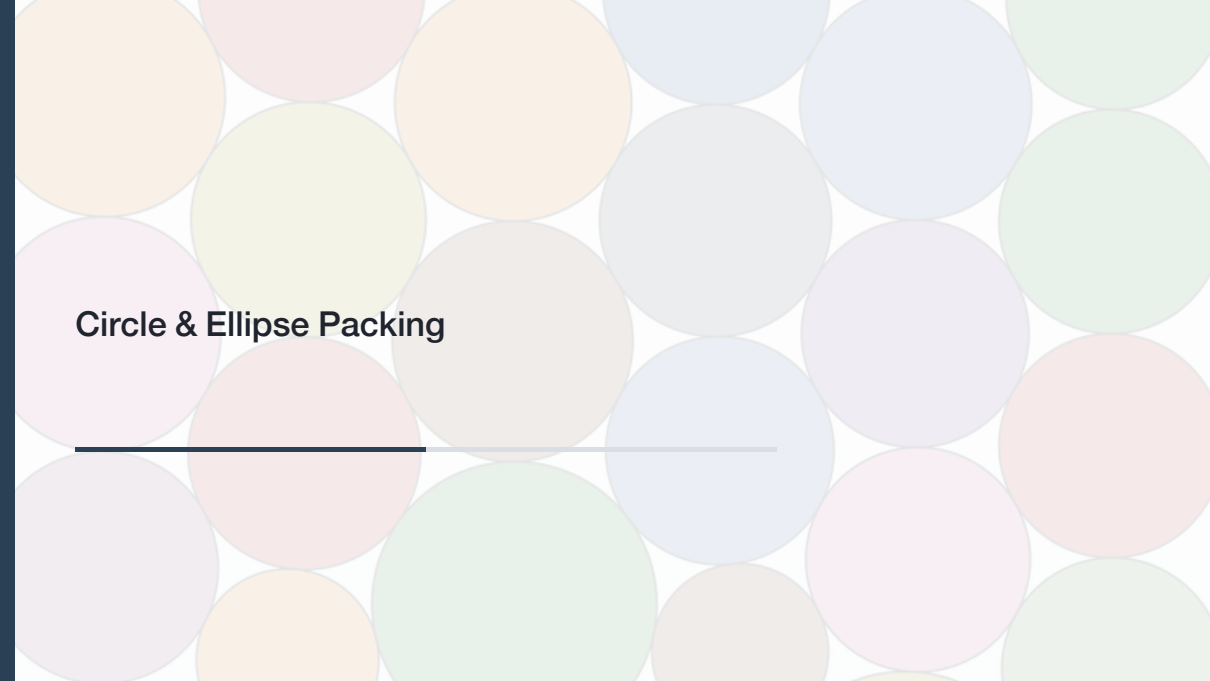
plato.asu.edu/ftp/mipfeas.html

Best open-source solver for MINLP

the engine behind the results in this talk

plato.asu.edu/ftp/minlp.html

Circle & Ellipse Packing

The background of the slide is a repeating pattern of overlapping circles. The circles are in various colors including shades of orange, yellow, green, blue, pink, and grey. They are arranged in a way that some circles are partially obscured by others, creating a dense, tessellated effect.

Circle Packing Model

Pack n non-overlapping circles into a unit square (or a rectangle of perimeter 4), maximizing the sum of radii

$$\max \sum_{i \in \mathcal{N}} r_i \quad (1)$$

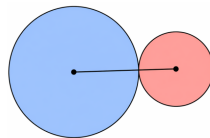
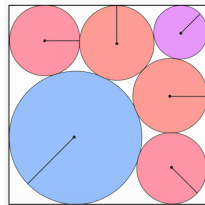
$$\text{s.t. } r_i \leq x_i \leq \alpha - r_i, \quad \text{for all } i \in \mathcal{N} \quad (2)$$

$$r_i \leq y_i \leq (2 - \alpha) - r_i, \quad \text{for all } i \in \mathcal{N} \quad (3)$$

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j)^2, \quad \text{for all } i, j \in \mathcal{N}, i < j \quad (4)$$

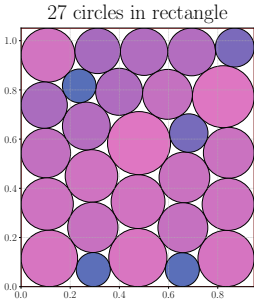
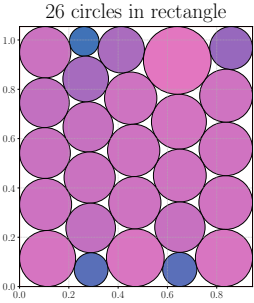
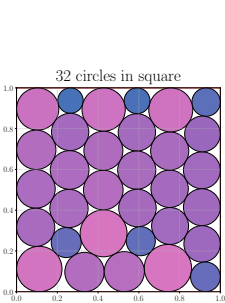
$$0 \leq r_i \leq \frac{\alpha}{2}, \quad \text{for all } i \in \mathcal{N} \quad (5)$$

$$0 < \alpha \leq 1 \quad (6)$$



New best-known solutions

Variant	n	Sum of radii	Previous best	Source
square	32	2.93 957	2.93794	(Georgiev et al., 2025; Novikov et al., 2025)
rectangle	26	2.63 930	2.638	https://erich-friedman.github.io/packing/cirRrec/
rectangle	27	2.6 9015	2.687	https://erich-friedman.github.io/packing/cirRrec/

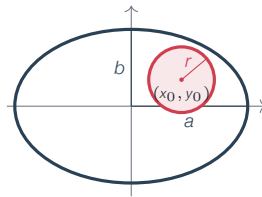


Circles in an Ellipse — the modeling challenge

- ◇ Pack n unit circles into an ellipse of minimum area (semi-axes a, b , centers x_i, y_i)
- ◇ How to write “circle inside ellipse” as algebraic constraints?

Contained $\iff$ no point is *in* the circle but *outside* the ellipse:

$$\begin{aligned} \nexists x, y \\ (x - x_0)^2 + (y - y_0)^2 \leq r^2 \\ b^2 x^2 + a^2 y^2 > a^2 b^2 \end{aligned}$$

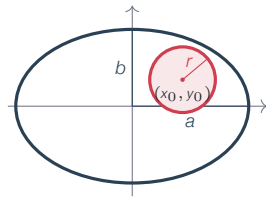


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S-Lemma (Yakubovich, 1971):

$$\exists t \geq 0 \forall x, y \in \mathbb{R} : \quad a^2 b^2 - b^2 x^2 - a^2 y^2 + t((x - x_0)^2 + (y - y_0)^2 - r^2) \geq 0$$

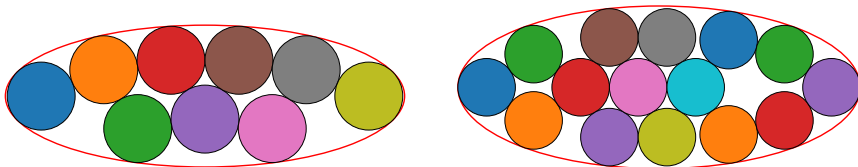
Circles in an Ellipse

$$\begin{array}{ll}
 & \min a \cdot b \cdot \pi \\
 \text{circles are disjoint} & \text{s.t. } (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4, \quad \forall i < j \\
 \text{circles are in the ellipse} & t_i - a^2 \geq 0, \quad \forall i \\
 & t_i - b^2 \geq 0, \quad \forall i \\
 & a^2 b^2 + t_i (x_i^2 + y_i^2 - 1) \geq 0, \quad \forall i \\
 & (t_i - a^2)(a^2 b^2 + t_i (x_i^2 + y_i^2 - 1)) - t_i^2 y_i^2 \geq 0, \quad \forall i \\
 & (t_i - b^2)(a^2 b^2 + t_i (x_i^2 + y_i^2 - 1)) - t_i^2 x_i^2 \geq 0, \quad \forall i \\
 & (t_i - a^2)(t_i - b^2)(a^2 b^2 + t_i (x_i^2 + y_i^2 - 1)) \\
 & \quad - t_i^2 (x_i^2 (t_i - a^2) + y_i^2 (t_i - b^2)) \geq 0, \quad \forall i \\
 \text{variable bounds} & t_i \geq 1, \quad \forall i \\
 & a, b \geq 1 \\
 \text{symmetry breaking} & a \geq b
 \end{array}$$

* Model has polynomial degree 8 (the $a^4 b^4$ term)

- ◇ All known solutions recovered
- ◇ **Two better solutions** for $n=9, 15$ — *different* configurations, not just slight tweaks

Size	New objective	Old objective	Source
9	12.2708π	12.403π	https://erich-friedman.github.io/packing/cirine1/
15	19.76329π	19.824π	https://erich-friedman.github.io/packing/cirine1/





Polygon & Platonic Solid Packing

Problem: pack regular m -gons into the smallest regular ℓ -gon

- ◇ Very well studied for squares into squares and triangles into triangles
- ◇ Challenge for modeling: Nonoverlapping circles are easy...
(shape literally defined equidistance, perfect rotational symmetry)
- ◇ ..., however, nonoverlapping polygons are harder, rotation comes into play



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Intersection between polygons empty?

- ◇ i.e., a system of inequalities does not have a solution?
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⇒ Farkas' Lemma!

Farkas' lemma: $Ax > b$ is infeasible $\iff \exists y \geq 0, y \neq 0$ with $y^\top A = 0$ and $y^\top b \geq 0$

Lemma (separation certificate)

Two convex m -gons with half-space representations $\{a_{i,k} x + b_{i,k} y \geq s_{i,k}\}$ and $\{a_{j,k} x + b_{j,k} y \geq s_{j,k}\}, k = 0, \dots, m-1$, have disjoint interiors if and only if there exist nonnegative multipliers $\lambda_1, \dots, \lambda_{2m}$ summing to one with

$$\begin{aligned}\sum_{k=0}^{m-1} \lambda_{k+1} a_{i,k} + \sum_{k=0}^{m-1} \lambda_{k+m+1} a_{j,k} &= 0 \\ \sum_{k=0}^{m-1} \lambda_{k+1} b_{i,k} + \sum_{k=0}^{m-1} \lambda_{k+m+1} b_{j,k} &= 0 \\ \sum_{k=0}^{m-1} \lambda_{k+1} s_{i,k} + \sum_{k=0}^{m-1} \lambda_{k+m+1} s_{j,k} &\geq 0\end{aligned}$$

Pack n regular m -gons into a regular ℓ -gon of minimum circumradius R :

$$\min R$$

$$\text{all inner vertices in outer } \ell\text{-gon} \quad R\rho_\ell + \sin(k\phi_\ell)(x_i + \sin(\theta_i + \delta_{m,j}))$$

$$+ \cos(k\phi_\ell)(y_i + \cos(\theta_i + \delta_{m,j})) \geq 0$$

$$\text{inner representation matches outer} \quad a_{i,j} = \sin(\theta_i + j\phi_m), \quad b_{i,j} = \cos(\theta_i + j\phi_m)$$

$$s_{i,j} = a_{i,j}x_i + b_{i,j}y_i - \rho_m$$

$$\text{polygons don't intersect (Farkas)} \quad \sum_{k=1}^{2m} \lambda_{i,j,k} = 1$$

$$\sum_{k=0}^{m-1} \lambda_{i,j,k+1} a_{i,k} + \sum_{k=0}^{m-1} \lambda_{i,j,k+m+1} a_{j,k} = 0$$

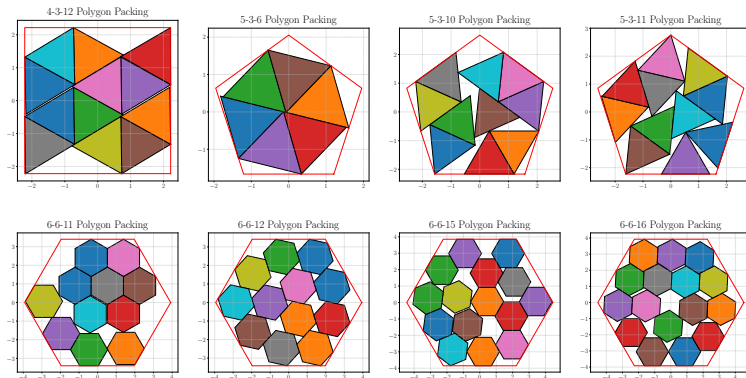
$$\sum_{k=0}^{m-1} \lambda_{i,j,k+1} b_{i,k} + \sum_{k=0}^{m-1} \lambda_{i,j,k+m+1} b_{j,k} = 0$$

$$\sum_{k=0}^{m-1} \lambda_{i,j,k+1} s_{i,k} + \sum_{k=0}^{m-1} \lambda_{i,j,k+m+1} s_{j,k} \geq 0$$

$$\text{symmetry breaking} \quad 0 \leq \theta_i \leq \phi_m, \quad \lambda_{i,j,k} \geq 0$$

$$\text{redundant, bound strengthening} \quad (x_i - x_j)^2 + (y_i - y_j)^2 \geq (2\rho_m)^2, \quad R \geq R_{\min}$$

Polygon Packing — 22 new best-known solutions



Improvements over decades-old best-known packings (Cantrell, Friedman, ...), and over some AlphaEvolve hexagon results.

All visualizations: <https://github.com/DominikKamp/Packing>

5 formulation variants tested systematically (750 polygon instances):

1. **Base** (Nodist): Farkas separation only
2. **+ Distance** (Dist): add redundant $(x_i - x_j)^2 + (y_i - y_j)^2 \geq (2\rho)^2$
 $\Rightarrow$ **best formulation**, dramatically better primal solutions
 $\Rightarrow$ SCIP 11 will make better use of them automatically (Hojny and Liberti, 2026)
3. **Inner separation**: replace Farkas by direct hyperplane constraints
 $\Rightarrow$ fewer variables but more nonlinearity, weaker results
4. **Farkas normalization**: per-polygon lower bounds on multipliers
 $\Rightarrow$ no improvement
5. **+ Symmetry breaking**: coordinate ordering + centroid constraints
 $\Rightarrow$ **hurts** heuristic performance

For details, see arxiv.org/abs/2605.04850

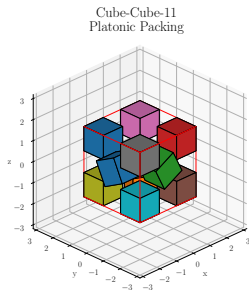
	Polygon (2D)	Platonic Solid (3D)
Position	(x_i, y_i)	(x_i, y_i, z_i)
Rotation	$\theta_i \in [0, \phi]$	$\theta_i, \iota_i, \kappa_i \in [0, 2\pi]$
Normals	$(a_{i,j}, b_{i,j})$ edge normals	$(a_{i,f}, b_{i,f}, c_{i,f})$ face normals
Separation	2 normal equations	3 normal equations
Multipliers	$2m$ per pair	$2 \mathcal{F} $ per pair

The Farkas certificate generalizes to 3D: edge normals $\rightarrow$ face normals, one rotation angle $\rightarrow$ a 3D rotation

- ◇ First systematic computational study

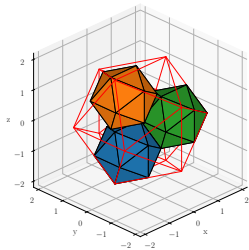
Table 1: Improving solution for packing 11 cubes into a cube

ℓ	m	n	Circumradius	Previous	Source
3	3	11	2.89445	2.91210	Friedman 1998

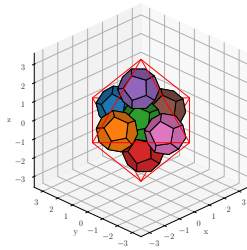


Some more 3D packings

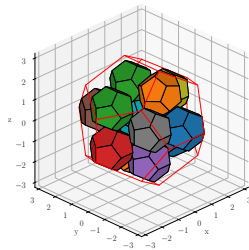
Icosahedron-Icosahedron-3
Platonic Packing



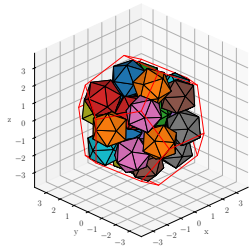
Octahedron-Dodecahedron-7
Platonic Packing



Dodecahedron-Dodecahedron-13
Platonic Packing



Dodecahedron-Icosahedron-22
Platonic Packing



And many more: <https://github.com/DominikKamp/Packing>

Take-aways

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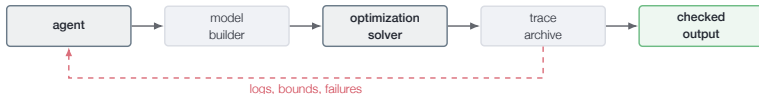
- ◇ Global nonlinear optimization has become a “model & solve” technology
- ◇ Off-the-shelf SCIP / Xpress simply solve problems other approaches struggle with:
Setting 35 records on packing problems (and counting)
- ◇ General solvers & models are very versatile
- ◇ Dual certificates aren't very strong here, but they come for free
- ◇ LLM results are impressive and open the door to new user groups:
“Claude, write me a model for...” and plugging that into a solver seems like a sweet spot

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Open question

How do we make LLMs more optimization-aware?



Agentic math-lab setup (Zimmer et al., 2026; Pelleriti et al., 2026).

Thank you!

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arxiv.org/abs/2605.04850



www.scipopt.org

- Timo Berthold, Dominik Kamp, Gioni Mexi, Sebastian Pokutta, and Imre Polik. Out-of-the-box global optimization for packing problems: New models and improved solutions, 2026a. URL <https://arxiv.org/abs/2605.04850>.
- Timo Berthold, Dominik Kamp, Gioni Mexi, Sebastian Pokutta, and Imre Pólik. Global optimization for combinatorial geometry problems revisited in the era of llms, 2026b. URL <https://arxiv.org/abs/2601.05943>.
- Bogdan Georgiev, Javier Gómez-Serrano, Terence Tao, and Adam Zsolt Wagner. Mathematical exploration and discovery at scale. *arXiv preprint arXiv:2511.02864*, 2025.
- Christopher Hojny and Leo Liberti. A computational comparison of handling distance constraints in minlp, 2026. URL <https://arxiv.org/abs/2605.02305>.
- Alexander Novikov, Ngán Vũ, Marvin Eisenberger, Emilien Dupont, Po-Sen Huang, Adam Zsolt Wagner, Sergey Shirobokov, Borislav Kozlovskii, Francisco JR Ruiz, Abbas Mehrabian, et al. Alphaevolve: A coding agent for scientific and algorithmic discovery. *arXiv preprint arXiv:2506.13131*, 2025.

Nico Pelleriti, Sree Harsha Nelaturu, Zhanke Zhou, Zongze Li, Max Zimmer, Bo Han, and Sebastian Pokutta. What do evolutionary coding agents evolve?, 2026. URL <https://arxiv.org/abs/2605.20086>. Preprint.

Max Zimmer, Nico Pelleriti, Christophe Roux, and Sebastian Pokutta. The agentic researcher: A practical guide to ai-assisted research in mathematics and machine learning. In *ICML 2026 Workshop on AI for Research (AI4Research)*, 2026. URL <https://openreview.net>. Oral presentation. OpenReview.